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Representations of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|4)$ and paraboson coherent states

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Abstract

We introduce and obtain multimode paraboson coherent states. In appropriate subspaces these coherent states provide a decomposition of unity where the measure, when expressed using the cat-type states, is positive definite. Bicoherent states where the mutually commuting lowering operators are diagonalized are also obtained. Matrix elements in the coherent state basis are calculated.

1 Introduction

Soon after parastatistics has been introduced [1], it was discovered that it has a deep algebraic structure. It turned out that any n pairs of parafermion operators generate the simple Lie algebra $\mathfrak{so}(2n+1)$ [2, 3], and n pairs of paraboson creation and annihilation operators $b_1^{\pm}, \ldots, b_n^{\pm}$ generate a Lie superalgebra [4], isomorphic to one of the basic classical Lie superalgebras in the classification of Kac [5], namely to the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|2n)$ [6]. Actually, the paraboson operators were introduced earlier by Wigner [7] in a search of the most general commutation relations between the position operator $\hat{q}$ and the momentum operator $\hat{p}$ of a one-dimensional oscillator, so that the Heisenberg equations are compatible with Hamilton's equations. The operators $\hat{p}$, $\hat{q}$ turned out to generate $\mathfrak{osp}(1|2)$ and Wigner was the first who found a class of (infinite-dimensional) representations of a Lie superalgebra [8]. Later on the results of Wigner gave rise to more general quantum systems (see [9] for references in this respect and for a general introduction to parastatistics), and in particular to Wigner quantum systems introduced by Palev [8, 10].

However only quite recently, the paraboson Fock spaces for the Lie superalgebra $\mathfrak{osp}(1|2n)$ were constructed [11]. These are lowest weight representations $V(p)$ characterized by a positive parameter p, called the order of the statistics. In [11], an explicit basis for the representation spaces $V(p)$ is introduced with the matrix elements of these representations. Because of the computational difficulties in the construction of the paraboson Fock spaces the paraboson coherent states (eigenstates of paraboson operators) were constructed only for one pair of paraboson operators [12]. In the present paper we use the results of [11] for the $n = 2$ case in order to obtain "coherent state" representations of two pairs of paraboson operators $b_1^{\pm}$, $b_2^{\pm}$. Coherent states play vital roles [13–16] in many contexts such as quantum optics, semiclassical quantization of systems with spin degrees of freedom, construction of quantum mechanical path integrals, the geometric quantization of coadjoint orbits, and so on. One specific motivation of our study lies in the realm of noncommutative spaces. It has been recently observed [17] that the fuzzy torus can be regarded as a single mode q-deformed parafermion. It is important to investigate higher order generalizations and supersymmetrization of such fuzzy spaces and their relations with n-body parastatistics. In this context the coherent states are expected to provide the star product structure that reflects the noncommutativity of such spaces, and allows building of quantum mechanical and field theoretical models on such spaces. Coherent state realization [18] of such star product, and its generalizations

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for noncommutative superspaces utilizing, for instance, the oscillator representation of $\mathfrak{osp}(1|2)$ coherent states have been achieved [19]. From these considerations the construction of the $\mathfrak{osp}(1|2n)$ coherent states based on n-body paraboson representations may have utility in analyzing fuzzy superspaces. Another motivation of the present work stems from the study of integrable models in quantum mechanics such as the Calogero model [20]. It has been observed [21], [22] that the single mode ($n = 1$) paraboson play a crucial role in understanding the Calogero model [20]. Our construction of the coherent states of the multi-mode parabosons, therefore, may facilitate a detailed construction of related quantum integrable models.

The plan of the present work is as follows. In section 2, we define the Lie superalgebra $\mathfrak{osp}(1|4)$ and give a description of its paraboson Fock representations $V(p)$. The b_1^--coherent states are constructed in section 3. Since the operators b_1^- and b_2^- do not commute but b_1^- and $(b_2^-)^2$ do, in section 4 we are dealing with the b_1^- and $(b_2^-)^2$-coherent states. Section 5 is devoted to the b_2^--matrix elements. The constructed coherent states allow the resolution of unity which is considered in section 6. We end the paper with some final remarks.

2 The Lie superalgebra $\mathfrak{osp}(1|4)$ and its paraboson representations

In matrix form the orthosymplectic Lie superalgebra $\mathfrak{osp}(1|4)$ [5] can be defined as a subset of all five by five matrices

$$\begin{pmatrix} 0 & a & b & c & d \\ c & e & f & x & y \\ d & g & h & y & z \\ -a & u & v & -e & -g \\ -b & v & w & -f & -h \end{pmatrix}, \tag{2.1}$$

where the nonzero entries are arbitrary complex numbers. The even subalgebra $\mathfrak{osp}(1|4)_{\bar{0}}$ consists of the matrices for which $a = b = c = d = 0$ and the odd subspace $\mathfrak{osp}(1|4)_{\bar{1}}$ of $\mathfrak{osp}(1|4)$ corresponds to the case in (2.1) when $e = f = g = h = x = y = z = u = v = w = 0$. Let the row and column indices run from 0 to 4 and denote by e_{ij} the matrix with zeros everywhere except a 1 on position (i, j). Then the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{osp}(1|4)$ is spanned by the diagonal elements

$$h_1 = e_{11} - e_{33}, \qquad h_2 = e_{22} - e_{44}. \tag{2.2}$$

In terms of the dual basis $\delta_1,\ \delta_2$ of $\mathfrak{h}^*$, the odd root vectors and corresponding roots of $\mathfrak{osp}(1|4)$ are given by:

$$\begin{aligned} &e_{0,k} - e_{k+2,0} \leftrightarrow -\delta_k, \qquad k = 1, 2, \\ &e_{0,k+2} + e_{k,0} \leftrightarrow \delta_k, \qquad k = 1, 2. \end{aligned}$$

The even root vectors and roots are

$$\begin{aligned} &e_{j,k} - e_{k+2,j+2} \leftrightarrow \delta_j - \delta_k, \qquad j \neq k = 1, 2, \\ &e_{j,k+2} + e_{k,j+2} \leftrightarrow \delta_j + \delta_k, \qquad j \leq k = 1, 2, \\ &e_{j+2,k} + e_{k+2,j} \leftrightarrow -\delta_j - \delta_k, \qquad j \leq k = 1, 2. \end{aligned}$$

We introduce the following multiples of the odd root vectors

$$b_k^+ = \sqrt{2}(e_{0,k+2} + e_{k,0}), \qquad b_k^- = \sqrt{2}(e_{0,k} - e_{k+2,0}) \qquad k = 1, 2. \tag{2.3}$$

Since all even root vectors can be obtained by anticommutators $\{b_j^\xi, b_k^\eta\}$, the following holds [6]

Theorem 1 (Ganchev and Palev) *As a Lie superalgebra defined by generators and relations,* $\mathfrak{osp}(1|4)$ *is generated by 4 odd elements* $b_k^{\pm}$ *subject to the following relations:*

$$[\{b_j^{\xi}, b_k^{\eta}\}, b_l^{\epsilon}] = (\epsilon - \xi)\delta_{jl} b_k^{\eta} + (\epsilon - \eta)\delta_{kl} b_j^{\xi}. \tag{2.4}$$

The operators b_j^+ are the positive odd root vectors, and the b_j^- are the negative odd root vectors. Relations (2.4) are the defining triple relations of the paraboson operators.

The paraboson Fock space $V(p)$ is characterized by $(j, k = 1, 2)$

$$(b_j^{\pm})^\dagger = b_j^{\mp}, \qquad b_j^- |0\rangle = 0, \qquad \{b_j^-, b_k^+\}|0\rangle = p\,\delta_{jk}\,|0\rangle. \tag{2.5}$$

Furthermore, it is easy to verify that

$$\{b_j^-, b_j^+\} = 2h_j \qquad (j = 1, 2). \tag{2.6}$$

Hence we have the following:

Corollary 2 *The paraboson Fock space* $V(p)$ *is the unitary irreducible representation (unirrep) of* $\mathfrak{osp}(1|4)$ *with lowest weight* $(\frac{p}{2}, \frac{p}{2})$.

In [11], an explicit basis and the matrix elements of these representations were constructed. We summarize the results:

Theorem 3 ([11]) *The* $\mathfrak{osp}(1|4)$ *representation* $V(p)$ *with lowest weight* $(\frac{p}{2}, \frac{p}{2})$ *is a unirrep if and only if* $p \geq 1$. *For* $p > 1$, *the representation space is spanned by the following orthonormal basis (Gelfand-Zetlin basis (GZ)):*

$$|m) = \left| \begin{array}{l} m_{12}, m_{22} \\ m_{11} \end{array} \right), \tag{2.7}$$

where (m_{12}, m_{22}) *is a partition* λ *of length at most 2, i.e.* m_{12} *and* m_{22} *are integers with*

$$m_{12} \geq m_{22} \geq 0, \quad \text{and} \quad m_{12} \geq m_{11} \geq m_{22}. \tag{2.8}$$

For $p = 1$, *the basis consists of all vectors (2.7)-(2.8) with* $m_{22} = 0$. *The explicit action of the* $\mathfrak{osp}(1|4)$ *generators in* $V(p)$ *is given by:*

$$\begin{aligned} b_1^+|m) =& \sqrt{m_{11} - m_{22} + 1}\; f_1(m_{12}, m_{22}) \left| \begin{array}{l} m_{12}+1, m_{22} \\ m_{11}+1 \end{array} \right) \\ &- \sqrt{m_{12} - m_{11}}\; f_2(m_{12}, m_{22}) \left| \begin{array}{l} m_{12}, m_{22}+1 \\ m_{11}+1 \end{array} \right), \end{aligned} \tag{2.9}$$

$$\begin{aligned} b_2^+|m) =& \sqrt{m_{12} - m_{11} + 1}\; f_1(m_{12}, m_{22}) \left| \begin{array}{l} m_{12}+1, m_{22} \\ m_{11} \end{array} \right) \\ &+ \sqrt{m_{11} - m_{22}}\; f_2(m_{12}, m_{22}) \left| \begin{array}{l} m_{12}, m_{22}+1 \\ m_{11} \end{array} \right), \end{aligned} \tag{2.10}$$

$$\begin{aligned} b_1^-|m) =& \sqrt{m_{11} - m_{22}}\; f_1(m_{12}-1, m_{22}) \left| \begin{array}{l} m_{12}-1, m_{22} \\ m_{11}-1 \end{array} \right) \\ &- \sqrt{m_{12} - m_{11} + 1}\; f_2(m_{12}, m_{22}-1) \left| \begin{array}{l} m_{12}, m_{22}-1 \\ m_{11}-1 \end{array} \right), \end{aligned} \tag{2.11}$$

$$\begin{aligned} b_2^-|m) =& \sqrt{m_{12} - m_{11}}\; f_1(m_{12}-1, m_{22}) \left| \begin{array}{l} m_{12}-1, m_{22} \\ m_{11} \end{array} \right) \\ &+ \sqrt{m_{11} - m_{22} + 1}\; f_2(m_{12}, m_{22}-1) \left| \begin{array}{l} m_{12}, m_{22}-1 \\ m_{11} \end{array} \right), \end{aligned} \tag{2.12}$$

$$h_1|m) = (\frac{p}{2} + m_{11})|m), \quad h_2|m) = (\frac{p}{2} + m_{12} + m_{22} - m_{11})|m) \tag{2.13}$$

where

$$f_1(m_{12},m_{22}) = (-1)^{m_{22}} \frac{(m_{12}+2+\mathcal{E}_{m_{12}}(p-2))^{1/2}}{(m_{12}-m_{22}+1+\mathcal{O}_{m_{12}-m_{22}})^{1/2}}, \tag{2.14}$$

$$f_2(m_{12},m_{22}) = \frac{(m_{22}+1+\mathcal{E}_{m_{22}}(p-2))^{1/2}}{(m_{12}-m_{22}+1-\mathcal{O}_{m_{12}-m_{22}})^{1/2}} \tag{2.15}$$

and the even *and* odd functions $\mathcal{E}_j$ *and* $\mathcal{O}_j$ *are defined by*

$$\begin{aligned} &\mathcal{E}_j = 1 \textit{ if } j \textit{ is even and 0 otherwise,} \\ &\mathcal{O}_j = 1 \textit{ if } j \textit{ is odd and 0 otherwise.} \end{aligned} \tag{2.16}$$

In the following sections, we shall assume that we are dealing with the generic case $p > 1$.

3 b_1^--coherent states

In this section we will construct coherent states of the operator b_1^- as eigenstates in $V(p)$

$$b_1^- \psi = \alpha \psi, \tag{3.1}$$

where α is a complex eigenvalue. Let $|\zeta\rangle \in V(p)$ be a weight vector annihilated by b_1^-, i.e.

$$h_1|\zeta\rangle = \zeta_1|\zeta\rangle, \;\; h_2|\zeta\rangle = \zeta_2|\zeta\rangle, \;\; b_1^-|\zeta\rangle = 0. \tag{3.2}$$

Lemma 4 *Let* $|\zeta\rangle \in V(p)$ *be a weight vector annihilated by* b_1^- *and let* $T_1 = b_1^- b_1^+ \in U(\mathfrak{osp}(1|4))$. *Then:*

- $$b_1^-(b_1^+)^n|\zeta\rangle = (n+\mathcal{O}_n(2\zeta_1-1))(b_1^+)^{n-1}|\zeta\rangle \tag{3.3}$$
- $$T_1(b_1^+)^n|\zeta\rangle = (n+1+\mathcal{E}_n(2\zeta_1-1))(b_1^+)^n|\zeta\rangle \tag{3.4}$$

For vectors v *in* $V(p)$ *which are* T_1*-eigenvectors with non-zero eigenvalue, i.e.* $T_1 v = \lambda v$*, we define* $T_1^{-1} v = \lambda^{-1} v$*. Then:*

- $$T_1^{-1}(b_1^+)^n|\zeta\rangle = (n+1+\mathcal{E}_n(2\zeta_1-1))^{-1}(b_1^+)^n|\zeta\rangle \tag{3.5}$$
- $$(b_1^+ T_1^{-1})^n|\zeta\rangle = \prod_{k=1}^{n}(k+\mathcal{O}_k(2\zeta_1-1))^{-1}(b_1^+)^n|\zeta\rangle \tag{3.6}$$

Proof. Equation (3.3) holds for $n = 1$:

$$b_1^- b_1^+|\zeta\rangle = \{b_1^-, b_1^+\}|\zeta\rangle = 2h_1|\zeta\rangle = 2\zeta_1|\zeta\rangle$$

and for $n = 2$:

$$b_1^-(b_1^+)^2|\zeta\rangle = [b_1^-, (b_1^+)^2]|\zeta\rangle = 2b_1^+|\zeta\rangle.$$

In the last expression we used the triple relation $[b_1^-, (b_1^+)^2] = 2b_1^+$ (see (2.4)). Now the result follows using induction on n:

$$\begin{aligned} b_1^-(b_1^+)^n|\zeta\rangle &= [b_1^-, (b_1^+)^2(b_1^+)^{n-2}]|\zeta\rangle \\ &= [b_1^-, (b_1^+)^2](b_1^+)^{n-2}|\zeta\rangle + (b_1^+)^2[b_1^-, (b_1^+)^{n-2}]|\zeta\rangle \\ &= (2(b_1^+)^{n-1} + (b_1^+)^2(n-2+\mathcal{O}_n(2\zeta_1-1))(b_1^+)^{n-3})|\zeta\rangle \\ &= (n+\mathcal{O}_n(2\zeta_1-1))(b_1^+)^{n-1})|\zeta\rangle. \end{aligned}$$

Formula (3.4) follows directly from (3.3). Because of the diagonal action of T_1 on weight vectors $(b_1^+)^n|\zeta\rangle$ of $V(p)$ and the fact that $n+1+\mathcal{O}_n(2\zeta_1-1)>0$ one concludes that (3.5) holds. Note that T_1^{-1} is not an element of the enveloping algebra; nevertheless its action on such vectors of $V(p)$ is well defined. The proof of (3.6) uses (3.5) and again induction.

□

The last result allows us to define a "vertex operator" $\chi(\alpha)$:

$$\chi(\alpha)=\sum_{n=0}^{\infty}\alpha^n(b_1^+T_1^{-1})^n=\frac{1}{1-\alpha b_1^+T_1^{-1}} \tag{3.7}$$

on vectors $|\zeta\rangle$ of the form (3.2). Then we have:

Lemma 5 *Let $|\zeta\rangle\in V(p)$ be a weight vector annihilated by b_1^- that is normalized (i.e. $\langle\zeta|\zeta\rangle=1$), and $\chi(\alpha)$ a vertex operator of the form (3.7). Then:*

- $\chi(\alpha)|\zeta\rangle\in V(p)$
- *The norm of $\chi(\alpha)|\zeta\rangle$ is given by*

$$\langle\,\chi(\alpha)|\zeta\rangle\mid\chi(\alpha)|\zeta\rangle\,\rangle={}_0F_1\left({-\atop \zeta_1};\left(\frac{\bar{\alpha}\alpha}{2}\right)^2\right)+\frac{\bar{\alpha}\alpha}{2\zeta_1}\,{}_0F_1\left({-\atop \zeta_1+1};\left(\frac{\bar{\alpha}\alpha}{2}\right)^2\right), \tag{3.8}$$

  where ${}_0F_1\left({-\atop a};x\right)$ is the classical hypergeometric series

$${}_0F_1\left({-\atop a};x\right)=\sum_{k=0}^{\infty}\frac{x^k}{(a)_k k!},\quad (a)_k=a(a+1)\cdots(a+k-1) \tag{3.9}$$

- $\chi(\alpha)|\zeta\rangle$ *is an eigenvector of b_1^- with eigenvalue α:*

$$b_1^-\chi(\alpha)|\zeta\rangle=\alpha\chi(\alpha)|\zeta\rangle. \tag{3.10}$$

Proof. The first assertion follows from (3.8), since it is sufficient to show that the norm of the vector is finite. Since vectors of different weights are orthogonal, one has

$$\langle\,\chi(\alpha)|\zeta\rangle\mid\chi(\alpha)|\zeta\rangle\,\rangle=\sum_{n=0}^{\infty}\bar{\alpha}^n\alpha^n\langle\,(b_1^+T_1^{-1})^n|\zeta\rangle\mid(b_1^+T_1^{-1})^n|\zeta\rangle\,\rangle.$$

Consider

$$\begin{aligned}
&\langle\,(b_1^+T_1^{-1})^{n+1}|\zeta\rangle\mid(b_1^+T_1^{-1})^{n+1}|\zeta\rangle\,\rangle\\
&=\langle\,b_1^+T_1^{-1}(b_1^+T_1^{-1})^n|\zeta\rangle\mid b_1^+T_1^{-1}(b_1^+T_1^{-1})^n|\zeta\rangle\,\rangle\\
&=\langle\,T_1^{-1}(b_1^+T_1^{-1})^n|\zeta\rangle\mid b_1^-b_1^+T_1^{-1}(b_1^+T_1^{-1})^n|\zeta\rangle\,\rangle\\
&=\langle\,T_1^{-1}(b_1^+T_1^{-1})^n|\zeta\rangle\mid(b_1^+T_1^{-1})^n|\zeta\rangle\,\rangle\\
&=(n+1+\mathcal{E}_n(2\zeta_1-1))^{-1}\langle\,(b_1^+T_1^{-1})^n|\zeta\rangle\mid(b_1^+T_1^{-1})^n|\zeta\rangle\,\rangle.
\end{aligned}$$

In the last expression we used (3.3) and (3.5). Now by induction it follows that

$$\langle\,(b_1^+T_1^{-1})^n|\zeta\rangle\mid(b_1^+T_1^{-1})^n|\zeta\rangle\,\rangle=\prod_{k=1}^{n}(k+\mathcal{O}_k(2\zeta_1-1))^{-1}. \tag{3.11}$$

Therefore

$$\langle\, \chi(\alpha)|\zeta\rangle \mid \chi(\alpha)|\zeta\rangle\,\rangle = \sum_{n=0}^{\infty} \bar{\alpha}^n \alpha^n \prod_{k=1}^{n} (k + \mathcal{O}_k(2\zeta_1 - 1))^{-1}$$
$$= 1 + \frac{(\frac{\bar{\alpha}\alpha}{2})^2}{(\zeta_1)1!} + \frac{(\frac{\bar{\alpha}\alpha}{2})^4}{(\zeta_1)(\zeta_1+1)2!} + \cdots$$
$$+ \frac{\bar{\alpha}\alpha}{2\zeta_1}\Big(1 + \frac{(\frac{\bar{\alpha}\alpha}{2})^2}{(\zeta_1+1)1!} + \frac{(\frac{\bar{\alpha}\alpha}{2})^4}{(\zeta_1+1)(\zeta_1+2)2!} + \cdots\Big)$$
$$= {}_0F_1\left(\begin{matrix} - \\ \zeta_1 \end{matrix}; \left(\frac{\bar{\alpha}\alpha}{2}\right)^2\right) + \frac{\bar{\alpha}\alpha}{2\zeta_1}\, {}_0F_1\left(\begin{matrix} - \\ \zeta_1 + 1 \end{matrix}; \left(\frac{\bar{\alpha}\alpha}{2}\right)^2\right).$$

Since the classical hypergeometric series (3.9) is convergent for any x one concludes $\chi(\alpha)|\zeta\rangle \in V(p)$.

The last part follows from the following computation:

$$b_1^- \chi(\alpha)|\zeta\rangle$$
$$= b_1^-\big(1 + \alpha b_1^+ T_1^{-1} + \alpha^2 (b_1^+ T_1^{-1})(b_1^+ T_1^{-1}) + \cdots\big)|\zeta\rangle$$
$$= \big(b_1^- + \alpha T_1 T_1^{-1} + \alpha^2 (T_1 T_1^{-1})(b_1^+ T_1^{-1}) + \cdots\big)|\zeta\rangle$$
$$= b_1^-|\zeta\rangle + \alpha\big(1 + \alpha(b_1^+ T_1^{-1}) + \alpha^2 (b_1^+ T_1^{-1})^2 + \cdots\big)|\zeta\rangle = \alpha\chi(\alpha)|\zeta\rangle.$$

□

The above considerations show that in order to construct b_1^--coherent states we must find a complete basis of the subspace of weight vectors of $V(p)$, annihilated by b_1^-. The weight of the vector $|m)$ is given by $(\frac{p}{2}, \frac{p}{2}) + (m_{11}, m_{12} + m_{22} - m_{11})$ (see (2.13)). Now if we consider the weights, one could construct vectors

$$|\zeta_{jk}\rangle = \sum_{i=0}^{j} c_i(j,k) \left| \begin{matrix} k+i, j-i \\ j \end{matrix} \right), \quad k = 0, 1, \cdots, \quad j = 0, 1, \cdots, k, \tag{3.12}$$

of weight $(\frac{p}{2}, \frac{p}{2}) + (j, k)$ with $b_1^-|\zeta_{jk}\rangle = 0$ and $\langle\zeta_{jk}|\zeta_{jk}\rangle = 1$ (in other words $\sum_{i=0}^{j} c_i(j,k)^2 = 1$). This construction is given by:

Proposition 6 *An orthonormal basis of the subspace of weight vectors of $V(p)$, annihilated by b_1^- is given by (3.12), where*

$$c_i(j,k) = \sqrt{\binom{k-j+i}{i}} \prod_{r=0}^{k-j} \sqrt{\frac{r + 1 + \mathcal{O}_r(p-2+2j)}{k + 1 - r + \mathcal{O}_{k-r}(p-2)}}$$
$$\times \prod_{s=1}^{i} (-1)^{j-s} \sqrt{\frac{(j + 1 - s + \mathcal{E}_{j-s}(p-2))(k - j + 2s + \mathcal{O}_{k+j-1})}{(k + 1 + s + \mathcal{E}_{k+s-1}(p-2))(k - j + 2s - \mathcal{O}_{k+j-1})}}. \tag{3.13}$$

Proof. The action of b_1^- on the GZ basis vectors gives

$$b_1^-|\zeta_{jk}\rangle = \sum_{i=0}^{j-1} \Big(c_{i+1}(j,k)\sqrt{i+1} f_1(k+i, j-i-1) - c_i(j,k)\sqrt{k+i-j+1} f_2(k+i, j-i-1)\Big)$$
$$\times \left| \begin{matrix} k+i, j-i-1 \\ j-1 \end{matrix} \right).$$

Therefore

$$c_{i+1}(j,k) = (-1)^{j-i-1}\sqrt{\frac{(k+i-j+1)(j-i+\mathcal{E}_{j-i-1}(p-2))(k-j+2i+2+\mathcal{O}_{k+j-1})}{(i+1)(k+i+2+\mathcal{E}_{k+i}(p-2))(k-j+2i+2-\mathcal{O}_{k+j-1})}}\, c_i(j,k).$$

Clearly, the coefficients $c_i(j,k)$ (see (3.13)) satisfy the last equation. The condition $\sum_{i=0}^{j} c_i(j,k)^2 = 1$ is equivalent to the following identity:

$$\sum_{i=0}^{j}\binom{k-j+i}{i}\prod_{r=1}^{i}\frac{(j+1-r+\mathcal{E}_{j-r}(p-2))(k-j+2r+\mathcal{O}_{k+j-1})}{(k+1+r+\mathcal{E}_{k+r-1}(p-2))(k-j+2r-\mathcal{O}_{k+j-1})}$$
$$= \prod_{r=0}^{k-j}\frac{(k+1-r+\mathcal{O}_{k-r}(p-2))}{(r+1+\mathcal{O}_r(p-2+2j))}. \tag{3.14}$$

We only sketch the proof of this identity. One has to consider four cases with j and k even or odd. In each of these cases the proof follows the following steps:

1) Consider the sum over i even and over i odd separately.
2) Rewrite the sums in hypergeometric form and use a summation theorem.
3) Combine and see that this is the right-hand side of (3.14).

To show that vectors of the form (3.12) form a basis of the space annihilated by b_1^-, one uses a weight argument and the explicit action of b_1^-, given by (2.11). Note that in $V(p)$, the multiplicity of the weight $(\frac{p}{2},\frac{p}{2})+(j,k)$ is given by $\min(j+1,k+1)$. For $k=0$, it follows from (2.11) that there is only one vector annihilated by b_1^-. For $k=1$, (2.11) and the above multiplicity allow the construction of only two vectors annihilated by b_1^-. More generally, the multiplicity argument and (2.11) yield at most $k+1$ vectors annihilated by b_1^- for a given k-value. Since all vectors (3.12) are linearly independent, the statement follows.

□

Combination of Proposition 6 and the previous lemma now yields the following result:

Proposition 7 *A complete (actually, an overcomplete) set of b_1^--coherent states $b_1^-\tilde\psi_{jk}(\alpha) = \alpha\tilde\psi_{jk}(\alpha)$ is defined by*

$$\tilde\psi_{jk}(\alpha) = \chi(\alpha)|\zeta_{jk}\rangle, \quad k=0,1,\cdots;\ j=0,1,\cdots,k, \tag{3.15}$$

where $\chi(\alpha)$ and $|\zeta_{jk}\rangle$ are given by (3.7) and (3.12)-(3.13) resp. and

$$\langle\,\tilde\psi_{jk}(\alpha)\mid\tilde\psi_{jk}(\alpha)\rangle = {}_0F_1\left(\begin{matrix}-\\ \frac{p}{2}+j\end{matrix};\left(\frac{\bar\alpha\alpha}{2}\right)^2\right) + \frac{\bar\alpha\alpha}{p+2j}\,{}_0F_1\left(\begin{matrix}-\\ \frac{p}{2}+j+1\end{matrix};\left(\frac{\bar\alpha\alpha}{2}\right)^2\right) \tag{3.16}$$

Proof. The only part left to be proved is (3.16). It follows directly from the fact that $T_1|\zeta_{jk}\rangle = 2h_1|\zeta_{jk}\rangle = (p+2j)|\zeta_{jk}\rangle$ and (3.8).

□

In a later section, it will be convenient to have this norm (3.16) expressed in a different way, using the modified Bessel function

$$I_\nu(x) = \sum_{n=0}^{\infty}\frac{(x/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}. \tag{3.17}$$

Comparing with (3.9) implies that (3.16) can be rewritten as

$$\mathcal{N}_{p,j,\alpha} = \langle\,\tilde\psi_{jk}(\alpha)\mid\tilde\psi_{jk}(\alpha)\rangle = \left(\frac{\bar\alpha\alpha}{2}\right)^{1-j-p/2}\Gamma(\frac{p}{2}+j)\left(I_{j-1+p/2}(\bar\alpha\alpha)+I_{j+p/2}(\bar\alpha\alpha)\right). \tag{3.18}$$

4 b_1^-- and $(b_2^-)^2$-coherent states

Using the defining triple paraboson relations (2.4) it is straightforward to see that the operators $(b_2^\pm)^2$ commute with b_1^- and b_1^+. Hence, the action of $(b_2^\pm)^2$ also commutes with T_1 and T_1^{-1}. Therefore one concludes that $(b_2^\pm)^2$ commutes with $\chi(\alpha)$:

$$(b_2^\pm)^2\tilde{\psi}_{jk}(\alpha) = \chi(\alpha)(b_2^\pm)^2|\zeta_{jk}\rangle. \tag{4.1}$$

First note that $(b_2^-)^2|\zeta_{jk}\rangle$ is a vector of weight $(\frac{p}{2},\frac{p}{2}) + (j, k-2)$ and second $b_1^-(b_2^-)^2|\zeta_{jk}\rangle = (b_2^-)^2 b_1^-|\zeta_{jk}\rangle = 0$. Since there is only one vector of weight $(\frac{p}{2}+j, \frac{p}{2}+k-2)$ annihilated by b_1^- one concludes that $(b_2^-)^2|\zeta_{jk}\rangle = c|\zeta_{j,k-2}\rangle$. We could find the constant c by computing $(b_2^-)^2|\zeta_{jk}\rangle$ on one of the GZ basis vectors of $|\zeta_{jk}\rangle$ and compare the result with the same GZ vector in $|\zeta_{j,k-2}\rangle$. The result follows

$$(b_2^-)^2|\zeta_{jk}\rangle = \sqrt{(k-1-j+\mathcal{E}_{k-j})(p+k-2+j+\mathcal{O}_{k+j})}|\zeta_{j,k-2}\rangle. \tag{4.2}$$

In a similar way one obtains

$$(b_2^+)^2|\zeta_{jk}\rangle = \sqrt{(k+1-j+\mathcal{E}_{k-j})(p+k+j+\mathcal{O}_{k+j})}|\zeta_{j,k+2}\rangle. \tag{4.3}$$

Therefore

$$(b_2^-)^2\tilde{\psi}_{jk}(\alpha) = \sqrt{(k-1-j+\mathcal{E}_{k-j})(p+k-2+j+\mathcal{O}_{k+j})}\tilde{\psi}_{j,k-2}(\alpha) \tag{4.4}$$

$$(b_2^+)^2\tilde{\psi}_{jk}(\alpha) = \sqrt{(k+1-j+\mathcal{E}_{k-j})(p+k+j+\mathcal{O}_{k+j})}\tilde{\psi}_{j,k+2}(\alpha). \tag{4.5}$$

Now it is not difficult to construct the *bicoherent* states which are common eigenstates of the mutually commuting b_1^- and $(b_2^-)^2$ operators:

$$b_1^-\Psi_{jl}(\alpha,\beta) = \alpha\Psi_{jl}(\alpha,\beta), \quad (b_2^-)^2\Psi_{jl}(\alpha,\beta) = \beta\Psi_{jl}(\alpha,\beta), \tag{4.6}$$

where

$$\Psi_{jl}(\alpha,\beta) = \sum_{k=0}^{\infty} \frac{\beta^{k+\lfloor \frac{l}{2}\rfloor}}{\sqrt{(2k)!!(p+2l)(p+2l+2)\cdots(p+2l+2(k-1))}}\tilde{\psi}_{j,2k+l}(\alpha), \tag{4.7}$$

$$j = 0,1,\ldots; \quad l = j, j+1.$$

The index jl in Ψ_{jl} refers to the weight of the lowest weight vector in the expansion of Ψ_{jl} in the GZ-basis (just as this was the case for $\tilde{\psi}_{jk}$). We will make a brief comment about certain potential utilities of the bicoherent states (4.7) in the Conclusion.

5 b_2^--matrix elements

In the previous section, (4.4) yields the matrix elements of $(b_2^-)^2$ for the set of b_1^--coherent states $\tilde{\psi}_{jk}(\alpha)$. The matrix elements of b_2^- for this set can also be computed. Let us consider the operator $\chi(\alpha)$ acting on a weight vector $|\zeta\rangle$ annihilated by b_1^-, and apply formula (3.6). Then one could

formally write

$$
\begin{aligned}
\chi(\alpha)|\zeta\rangle &= \sum_{n=0}^{\infty} \alpha^n (b_1^+ T_1^{-1})^n |\zeta\rangle \\
&= \sum_{n=0}^{\infty} \alpha^{2n} (b_1^+ T_1^{-1})^{2n} |\zeta\rangle + \sum_{n=0}^{\infty} \alpha^{2n+1} (b_1^+ T_1^{-1})^{2n+1} |\zeta\rangle \\
&= \sum_{n=0}^{\infty} \frac{1}{n!(\zeta_1)_n} \Big(\frac{\alpha b_1^+}{2}\Big)^{2n} |\zeta\rangle + \frac{\alpha b_1^+}{2\zeta_1} \sum_{n=0}^{\infty} \frac{1}{n!(\zeta_1+1)_n} \Big(\frac{\alpha b_1^+}{2}\Big)^{2n} |\zeta\rangle \\
&= {}_0F_1\left({- \atop \zeta_1}; \Big(\frac{\alpha b_1^+}{2}\Big)^2 \right) |\zeta\rangle + \frac{\alpha b_1^+}{2\zeta_1}\, {}_0F_1\left({- \atop \zeta_1+1}; \Big(\frac{\alpha b_1^+}{2}\Big)^2 \right) |\zeta\rangle. \qquad (5.1)
\end{aligned}
$$

Now we compute b_2^--matrix elements for the coherent states. Note that, by a weight argument, $\langle\tilde\psi_{j',k'}(\alpha')|b_2^-|\tilde\psi_{jk}(\alpha)\rangle$ can be nonzero only if $k' = k-1$. First, use (5.1) and the fact that b_2^- commutes with $(b_1^+)^2$:

$$
\begin{aligned}
&\langle\tilde\psi_{j',k-1}(\alpha')|b_2^-|\tilde\psi_{jk}(\alpha)\rangle \\
&= \langle\tilde\psi_{j',k-1}(\alpha')|b_2^- \left({}_0F_1\left({- \atop \frac{p}{2}+j}; \Big(\frac{\alpha b_1^+}{2}\Big)^2 \right) + \frac{\alpha b_1^+}{p+2j}\, {}_0F_1\left({- \atop \frac{p}{2}+j+1}; \Big(\frac{\alpha b_1^+}{2}\Big)^2 \right) \right) |\zeta_{jk}\rangle \\
&= \langle\tilde\psi_{j',k-1}(\alpha')|\, {}_0F_1\left({- \atop \frac{p}{2}+j}; \Big(\frac{\alpha b_1^+}{2}\Big)^2 \right) b_2^- |\zeta_{jk}\rangle + \langle\psi_{j',k-1}(\alpha')|\, {}_0F_1\left({- \atop \frac{p}{2}+j+1}; \Big(\frac{\alpha b_1^+}{2}\Big)^2 \right) \frac{\alpha b_2^- b_1^+}{p+2j} |\zeta_{jk}\rangle.
\end{aligned}
$$

Now, use the action of b_1^+ to the left and the action $b_1^-\tilde\psi_{j,k-1}(\alpha') = \alpha'\tilde\psi_{j,k-1}(\alpha')$. This yields:

$$
\begin{aligned}
&\langle\tilde\psi_{j',k-1}(\alpha')|b_2^-|\tilde\psi_{jk}(\alpha)\rangle \\
&= {}_0F_1\left({- \atop \frac{p}{2}+j}; \Big(\frac{\alpha\bar\alpha'}{2}\Big)^2 \right) \langle\tilde\psi_{j',k-1}(\alpha')|b_2^-|\zeta_{jk}\rangle + {}_0F_1\left({- \atop \frac{p}{2}+j+1}; \Big(\frac{\alpha\bar\alpha'}{2}\Big)^2 \right) \frac{\alpha}{p+2j} \langle\tilde\psi_{j',k-1}(\alpha')|b_2^- b_1^+|\zeta_{jk}\rangle.
\end{aligned}
$$

So the computation is reduced to computing the above two matrix elements. Using the explicit form of $|\zeta_{jk}\rangle$, the action of b_1^+ and b_2^- on GZ-basis elements, and the expansion of $\tilde\psi_{j',k-1}(\alpha')$ in terms of GZ-basis elements one finds:

$$
\langle\tilde\psi_{j',k-1}(\alpha')|b_2^-|\zeta_{jk}\rangle = \begin{cases}
\dfrac{p-2}{p-2+2j}\sqrt{k-j+\mathcal{O}_{k-j}(p-1+2j)} & \text{if } j'=j \\[2ex]
2(-1)^{j-1}\bar\alpha' \dfrac{\sqrt{j(p-2+j)(p+k+j-1-\mathcal{E}_{k-j})}}{(p+2j-2)^{3/2}} & \text{if } j'=j-1 \\[2ex]
0 & \text{otherwise}
\end{cases}
$$

$$
\langle\tilde\psi_{j',k-1}(\alpha')|b_2^- b_1^+|\zeta_{jk}\rangle = \begin{cases}
-\bar\alpha' \dfrac{p-2}{p+2j}\sqrt{k-j+\mathcal{O}_{k-j}(p-1+2j)} & \text{if } j'=j \\[2ex]
2(-1)^{j}\sqrt{\dfrac{(j+1)(p-1+j)(k-j-\mathcal{O}_{k-j})}{(p+2j)}} & \text{if } j'=j+1 \\[2ex]
0 & \text{otherwise}
\end{cases}
$$

Hence $\langle\tilde\psi_{j',k-1}(\alpha')|b_2^-|\tilde\psi_{jk}(\alpha)\rangle$ is 0 for $j'\neq j-1,j,j+1$. In the other cases it is given by:

$$\begin{aligned}
&\langle\tilde\psi_{j-1,k-1}(\alpha')|b_2^-|\tilde\psi_{jk}(\alpha)\rangle\\
&= {}_0F_1\left({- \atop \frac{p}{2}+j};\left(\frac{\alpha\bar\alpha'}{2}\right)^2\right)2(-1)^{j-1}\bar\alpha'\,\frac{\sqrt{j(p-2+j)(p+k+j-1-\mathcal{E}_{k-j})}}{(p+2j-2)^{3/2}},
\end{aligned}$$

$$\begin{aligned}
&\langle\tilde\psi_{j,k-1}(\alpha')|b_2^-|\tilde\psi_{jk}(\alpha)\rangle\\
&= {}_0F_1\left({- \atop \frac{p}{2}+j};\left(\frac{\alpha\bar\alpha'}{2}\right)^2\right)\frac{p-2}{p-2+2j}\sqrt{k-j+\mathcal{O}_{k-j}(p-1+2j)}\\
&- {}_0F_1\left({- \atop \frac{p}{2}+j+1};\left(\frac{\alpha\bar\alpha'}{2}\right)^2\right)\alpha\bar\alpha'\frac{p-2}{(p+2j)^2}\sqrt{k-j+\mathcal{O}_{k-j}(p-1+2j)},
\end{aligned}$$

$$\begin{aligned}
&\langle\tilde\psi_{j+1,k-1}(\alpha')|b_2^-|\tilde\psi_{jk}(\alpha)\rangle\\
&= {}_0F_1\left({- \atop \frac{p}{2}+j+1};\left(\frac{\alpha\bar\alpha'}{2}\right)^2\right)2\alpha(-1)^j\,\frac{\sqrt{(j+1)(p-1+j)(k-j-\mathcal{O}_{k-j})}}{(p+2j)^{3/2}}.
\end{aligned}$$

6 Resolution of the identity via $|\psi_{jk}(\alpha)\rangle$ states

We now discuss the resolution of the identity operator via the b_1^--coherent states $\tilde\psi_{jk}(\alpha)$. To be precise, we restrict ourselves to the states obtained by repeated actions of b_1^+ on the vector $|\zeta_{jk}\rangle$ ($k=0,1,\ldots;\ j=0,1,\ldots,k$) that is annihilated by b_1^-:

$$|\zeta_{jk};n\rangle=\frac{(b_1^+)^n}{\sqrt{2^n\left(\frac{n-\mathcal{O}_n}{2}\right)!\left(\frac{p}{2}+j\right)_{\frac{n+\mathcal{O}_n}{2}}}}\,|\zeta_{jk}\rangle,\qquad n=0,1,2,\ldots. \tag{6.1}$$

The action of $b_1^\pm$ operators on these orthonormal states of weights $(\frac{p}{2}+j+n,\frac{p}{2}+k)$ are given by

$$\begin{aligned}
b_1^+|\zeta_{jk};n\rangle &= \sqrt{(p+2j)\mathcal{E}_n+n+\mathcal{O}_n}\,|\zeta_{jk};n+1\rangle,\\
b_1^-|\zeta_{jk};n+1\rangle &= \sqrt{(p+2j)\mathcal{E}_n+n+\mathcal{O}_n}\,|\zeta_{jk};n\rangle.
\end{aligned}$$

In the following we consider the subspace $\mathcal{V}_{\langle j\rangle,k}$ spanned by all vectors $|\zeta_{jk};n\rangle$ with $n=0,1,2,\ldots$. For our discussion of the resolution of unity the state vectors $|\zeta_{jk};n\rangle$ and the subspace $\mathcal{V}_{\langle j\rangle,k}$ play a vital role.

Let us now consider the *normalized* b_1^--coherent states determined by (3.16) and (3.18):

$$|\psi_{jk}(\alpha)\rangle=\mathcal{N}_{p,j,\alpha}^{-1/2}\,\tilde\psi_{jk}(\alpha)=\mathcal{N}_{p,j,\alpha}^{-1/2}\,\chi(\alpha)\,|\zeta_{jk}\rangle. \tag{6.2}$$

Employing (5.1) and (6.1) the normalized coherent states $|\psi_{jk}(\alpha)\rangle$ may be expanded in the discrete basis $|\zeta_{jk};n\rangle$ of the subspace $\mathcal{V}_{\langle j\rangle,k}$ as

$$\begin{aligned}
|\psi_{jk}(\alpha)\rangle &= \mathcal{N}_{p,j,\alpha}^{-1/2}\left[{}_0F_1\left({- \atop \frac{p}{2}+j};\left(\frac{\alpha b_1^+}{2}\right)^2\right)+\frac{\alpha b_1^+}{p+2j}\,{}_0F_1\left({- \atop \frac{p}{2}+j+1};\left(\frac{\alpha b_1^+}{2}\right)^2\right)\right]|\zeta_{jk}\rangle\\
&= \left(\left(\frac{\bar\alpha\alpha}{2}\right)^{1-\frac{p}{2}-j}\left(I_{\frac{p}{2}+j-1}(\bar\alpha\alpha)+I_{\frac{p}{2}+j}(\bar\alpha\alpha)\right)\right)^{-1/2}\\
&\times\sum_{n=0}^{\infty}\Big(\frac{\alpha^{2n}}{\sqrt{n!\,2^{2n}\,\Gamma\left(\frac{p}{2}+j+n\right)}}\,|\zeta_{jk};2n\rangle+\frac{\alpha^{2n+1}}{\sqrt{n!\,2^{2n+1}\,\Gamma\left(\frac{p}{2}+j+n+1\right)}}\,|\zeta_{jk};2n+1\rangle\Big).
\end{aligned}\tag{6.3}$$

Being an overcomplete set the distinct normalized coherent states are not orthogonal:

$$\langle\psi_{jk}(\alpha')|\psi_{jk}(\alpha)\rangle = \frac{\left(\frac{\bar{\alpha'}\alpha}{2}\right)^{1-\frac{p}{2}-j}\left(I_{\frac{p}{2}+j-1}(\bar{\alpha'}\alpha)+I_{\frac{p}{2}+j}(\bar{\alpha'}\alpha)\right)}{\left[\left(\frac{|\alpha'|^2\,|\alpha|^2}{4}\right)^{1-\frac{p}{2}-j}\left(I_{\frac{p}{2}+j-1}(|\alpha'|^2)+I_{\frac{p}{2}+j}(|\alpha'|^2)\right)\left(I_{\frac{p}{2}+j-1}(|\alpha|^2)+I_{\frac{p}{2}+j}(|\alpha|^2)\right)\right]^{1/2}}. \tag{6.4}$$

The reflection property of the modified Bessel function $I_\nu(-x) = (-1)^\nu\, I_\nu(x)$ requires the following inner product to be real:

$$\langle\psi_{jk}(-\alpha)|\psi_{jk}(\alpha)\rangle = \langle\psi_{jk}(\alpha)|\psi_{jk}(-\alpha)\rangle = \frac{I_{\frac{p}{2}+j-1}(|\alpha|^2)-I_{\frac{p}{2}+j}(|\alpha|^2)}{I_{\frac{p}{2}+j-1}(|\alpha|^2)+I_{\frac{p}{2}+j}(|\alpha|^2)}. \tag{6.5}$$

We note that the reality of the overlap function (6.5) allows us to construct a cat-type two-dimensional subspace with orthogonal bases:

$$|\psi_{jk}(\alpha)\rangle_\pm = \frac{|\psi_{jk}(\alpha)\rangle \pm \psi_{jk}(-\alpha)\rangle}{||\psi_{jk}(\alpha)\rangle \pm |\psi_{jk}(-\alpha)\rangle|}, \qquad {}_+\langle\psi_{jk}(\alpha)|\psi_{jk}(\alpha)\rangle_- = 0, \tag{6.6}$$

where the normalized states $|\psi_{jk}(\alpha)\rangle_\pm$ explicitly read

$$|\psi_{jk}(\alpha)\rangle_+ = \left(\left(\frac{|\alpha|^2}{2}\right)^{1-p/2-j} I_{\frac{p}{2}+j-1}(|\alpha|^2)\right)^{-1/2}\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{\sqrt{n!\,2^{2n}\,\Gamma\left(\frac{p}{2}+j+n\right)}}\,|\zeta_{jk};2n\rangle, \tag{6.7}$$

$$|\psi_{jk}(\alpha)\rangle_- = \left(\left(\frac{|\alpha|^2}{2}\right)^{1-\frac{p}{2}-j} I_{\frac{p}{2}+j}(|\alpha|^2)\right)^{-1/2}\sum_{n=0}^{\infty}\frac{\alpha^{2n+1}}{\sqrt{n!\,2^{2n+1}\,\Gamma\left(\frac{p}{2}+j+n+1\right)}}\,|\zeta_{jk};2n+1\rangle. \tag{6.8}$$

Following [23] we now provide a resolution of the identity operator on the subspace $\mathcal{V}_{\langle j\rangle,k}$ via the coherent states $|\psi_{jk}(\alpha)\rangle$. To proceed, using the completeness of the discrete orthonormal basis states on the weight space $\mathcal{V}_{\langle j\rangle,k}$, one can write

$$\sum_{n=0}^{\infty}|\zeta_{jk};n\rangle\langle\zeta_{jk};n| = \mathbb{I}, \tag{6.9}$$

where $\mathbb{I}$ stands for the identity operator on $\mathcal{V}_{\langle j\rangle,k}$. Using the polar decomposition of the complex plane

$$\alpha = \rho\,\exp(i\theta), \qquad d^2\alpha = \frac{\rho\,d\rho\,d\theta}{2\pi}$$

concurrently with our construction (6.3) of the coherent states $|\psi_{jk}(\alpha)\rangle$ we integrate on the angular variable to obtain

$$\begin{aligned}\int_0^{2\pi}\frac{d\theta}{2\pi}\,|\psi_{jk}(\alpha)\rangle\langle\psi_{jk}(\alpha)| &= \left(\frac{\rho^2}{2}\right)^{\frac{p}{2}+j-1}\left(I_{\frac{p}{2}+j-1}(\rho^2)+I_{\frac{p}{2}+j}(\rho^2)\right)^{-1}\\ &\quad\times\sum_{n=0}^{\infty}\Big(\frac{1}{n!\,\Gamma\left(\frac{p}{2}+j+n\right)}\left(\frac{\rho^2}{2}\right)^{2n}|\zeta_{jk};2n\rangle\,\langle\zeta_{jk};2n|\\ &\quad+\frac{1}{n!\,\Gamma\left(\frac{p}{2}+j+n+1\right)}\left(\frac{\rho^2}{2}\right)^{2n+1}|\zeta_{jk};2n+1\rangle\,\langle\zeta_{jk};2n+1|\Big).\end{aligned} \tag{6.10}$$

We observe that in order to construct a resolution of unity on the current subspace it is necessary to consider the off-diagonal elements $|\psi_{jk}(\alpha)\rangle\,\langle\psi_{jk}(-\alpha)|$ of the density matrix. Using as yet to be

determined functions $F_I(\rho)$, $F_{II}(\rho)$ over the entire complex α plane, we employ (6.10) to obtain

$$\int d^2\alpha \Big(\frac{\rho^2}{2}\Big)^{1-\frac{p}{2}-j} \Big(I_{\frac{p}{2}+j-1}(\rho^2) + I_{\frac{p}{2}+j}(\rho^2)\Big)\, F_I(\rho)\times$$
$$\frac{1}{2}\Big(|\psi_{jk}(\alpha)\rangle\, \langle\psi_{jk}(\alpha)| + |\psi_{jk}(\alpha)\rangle\, \langle\psi_{jk}(-\alpha)|\Big) = \sum_{n=0}^{\infty} |\zeta_{jk}; 2n\rangle\, \langle\zeta_{jk}; 2n|, \tag{6.11}$$
$$\int d^2\alpha \Big(\frac{\rho^2}{2}\Big)^{1-\frac{p}{2}-j} \Big(I_{\frac{p}{2}+j-1}(\rho^2) + I_{\frac{p}{2}+j}(\rho^2)\Big)\, F_{II}(\rho)\times$$
$$\frac{1}{2}\Big(|\psi_{jk}(\alpha)\rangle\, \langle\psi_{jk}(\alpha)| - |\psi_{jk}(\alpha)\rangle\, \langle\psi_{jk}(-\alpha)|\Big) = \sum_{n=0}^{\infty} |\zeta_{jk}; 2n+1\rangle\, \langle\zeta_{jk}; 2n+1|. \tag{6.12}$$

The functions $F_I(\rho)$, $F_{II}(\rho)$ introduced above are required to satisfy the Stieltjes moment relations

$$\int_0^\infty d\rho\, \rho^{4n+1}\, F_I(\rho) = 2^{2n}\, \Gamma(n+1)\, \Gamma\Big(\frac{p}{2} + j + n\Big), \tag{6.13}$$
$$\int_0^\infty d\rho\, \rho^{4n+3}\, F_{II}(\rho) = 2^{2n+1}\, \Gamma(n+1)\, \Gamma\Big(\frac{p}{2} + j + n + 1\Big). \tag{6.14}$$

These functions are now explicitly determined by constructing the inverse Mellin transform. Using analytic continuation methods we express them as

$$F_I(\rho) = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} dz\, \Big(\frac{\rho^2}{2}\Big)^{-2z}\, \Gamma\Big(z + \frac{1}{2}\Big)\, \Gamma\Big(z + \frac{p}{2} + j - \frac{1}{2}\Big), \tag{6.15}$$
$$F_{II}(\rho) = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} dz\, \Big(\frac{\rho^2}{2}\Big)^{-2z}\, \Gamma(z)\, \Gamma\Big(z + \frac{p}{2} + j\Big), \tag{6.16}$$

where the pole structure of $\Gamma(z)$ on the negative real axis reads

$$\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\, \Big(\frac{1}{\epsilon} + \psi(n+1) + O(\epsilon)\Big), \qquad \psi(z) = (\ln\Gamma(z))'. \tag{6.17}$$

The contour integrals listed in (6.15) and (6.16) have distinct singularity structures for even integral values of p, and for the generic case $p > 1$. We first construct the inverse transforms for the even integral values: $p = 2m$, $m = 1, 2, \ldots$. Substituting the Laurent expansion (6.17) in (6.15) it is evident that the integrand in (6.15) has $m + j - 1$ simple poles at $z = -n - 1/2$, $n = 0, 1, \ldots, (m + j - 2)$, and an infinite number of poles of order 2 at $z = 1/2 - m - j - n$, $n = 0, 1, 2, \ldots$. The integral vanishes exponentially as $|z| \to \infty$ on the left half-plane. Adjoining the contour in (6.15) with a semicircle $|z| = R$ on the left half-plane, and then proceeding to the limiting value of its radius, $R \to \infty$, we get the integral (6.15) as a sum of the contributions arising from the simple and double poles, given, respectively, as follows:

$$2 \sum_{n=0}^{m+j-2} \frac{(-1)^n}{n!}\, \Big(\frac{\rho^2}{2}\Big)^{2n+1}\, (m + j - n - 2)!\,, \tag{6.18}$$

$$2\,(-1)^{m+j-1} \sum_{n=0}^{\infty} \frac{1}{n!\,(m+j+n-1)!}\, \Big(\frac{\rho^2}{2}\Big)^{p+2j+2n-1}\, \Big(\psi(m+j+n) + \psi(n+1) - 2\, \ln\Big(\frac{\rho^2}{2}\Big)\Big). \tag{6.19}$$

Combining the above contributions of the residues we obtain the promised explicit expression of the measure:

$$F_I(\rho) = 4\Big(\frac{\rho^2}{2}\Big)^{m+j}\, K_{m+j-1}(\rho^2), \tag{6.20}$$

where the modified Bessel function of the second kind $K_\nu(z)$ for an integral order ν is given by

$$K_\nu(z) = \frac{1}{2}\sum_{n=0}^{\nu-1}(-1)^n\,\frac{(\nu-n-1)!}{n!}\,\big(\tfrac{z}{2}\big)^{2n-\nu}$$
$$+\,(-1)^{\nu+1}\sum_{n=0}^{\infty}\frac{1}{n!(\nu+n)!}\Big(\ln\big(\tfrac{z}{2}\big)-\frac{1}{2}\psi(n+1)-\frac{1}{2}\psi(\nu+n+1)\Big). \tag{6.21}$$

To evaluate the contour integral (6.16) for the even integral values $p = 2\,m$, $m = 1, 2, \dots$ we notice, as before, that the integrand has $m + j$ simple poles at $z = -n$, $n = 0, 1, \dots, m + j - 1$, and an infinite number of poles of order 2 at $z = -m-j-n$, $n = 0, 1, 2, \dots$, respectively. The contribution of the residues at these poles now yields the corresponding measure function:

$$F_{II}(\rho) = 4\Big(\frac{\rho^2}{2}\Big)^{m+j}\,K_{m+j}(\rho^2). \tag{6.22}$$

We now turn to the case of *generic* values of the order parameter $p > 1$. Contrary to the earlier instance of even integral values of p, now the contour integral (6.15) of the inverse Mellin transform has two infinite sequences of simple poles at $z = -\frac{1}{2} - n$, $n = 0, 1, 2, \dots$, and $z = \frac{1}{2} - \frac{p}{2} - j - n$, $n = 0, 1, 2, \dots$. The corresponding residues to the contour integral (6.15) read

$$2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\Big(\frac{\rho^2}{2}\Big)^{2n+1}\,\Gamma\Big(\frac{p}{2}+j-n-1\Big), \tag{6.23}$$
$$2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\Big(\frac{\rho^2}{2}\Big)^{p+2j+2n-1}\,\Gamma\Big(1-\frac{p}{2}-j-n\Big), \tag{6.24}$$

respectively. Employing the reflection property of the gamma functions

$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin\pi z}$$

the residues (6.23) and (6.24) may be summed to yield the measure

$$F_I(\rho) = \frac{2\pi}{\sin\,\pi(\frac{p}{2}+j)}\,\Big(\frac{\rho^2}{2}\Big)^{\frac{p}{2}+j}\,\Big(I_{\frac{p}{2}+j-1}(\rho^2) - I_{1-\frac{p}{2}-j}(\rho^2)\Big)$$
$$= 4\Big(\frac{\rho^2}{2}\Big)^{\frac{p}{2}+j}\,K_{\frac{p}{2}+j-1}(\rho^2). \tag{6.25}$$

In the second equality in (6.25) we used the general construction of the modified Bessel function of the second kind $K_\nu(z)$ of arbitrary order ν:

$$K_\nu(z) = \frac{\pi}{2\,\sin(\nu\,\pi)}\,\big(I_{-\nu}(z) - I_\nu(z)\big). \tag{6.26}$$

Comparing the measure function (6.25) with the case (6.20) for even integral values of p, we observe that the form (6.25) is valid for all values of $p > 1$. Proceeding as before we observe that the integrand in (6.16) has two sequences of simple poles at $z = -n$, $n = 0, 1, 2, \dots$, and $z = -\frac{p}{2} - j - n$, $n = 0, 1, 2, \dots$, respectively. Evaluating their contributions we produce the measure function $F_{II}(\rho)$:

$$F_{II}(\rho) = \frac{2\pi}{\sin\,\pi(\frac{p}{2}+j)}\,\Big(\frac{\rho^2}{2}\Big)^{\frac{p}{2}+j}\,\Big(I_{-\frac{p}{2}-j}(\rho^2) - I_{\frac{p}{2}+j}(\rho^2)\Big)$$
$$= 4\Big(\frac{\rho^2}{2}\Big)^{\frac{p}{2}+j}\,K_{\frac{p}{2}+j}(\rho^2). \tag{6.27}$$

Comparison with (6.22) again reveals that the form (6.27) of the measure is universally true for arbitrary $p > 1$.

The coherent states $|\psi_{jk}(\alpha)\rangle$ now provide a decomposition of the identity operator in the subspace $\mathcal{V}_{\langle j \rangle,k}$ with an explicitly known weight function. Combining (6.11) and (6.25) the projection operator on the even subspace may be realized as

$$\begin{aligned} &\int d^2\alpha\, \rho^2 \left(I_{\frac{p}{2}+j-1}(\rho^2) + I_{\frac{p}{2}+j}(\rho^2) \right) K_{\frac{p}{2}+j-1}(\rho^2) \Big(|\psi_{jk}(\alpha)\rangle \langle \psi_{jk}(\alpha)| + |\psi_{jk}(\alpha)\rangle \langle \psi_{jk}(-\alpha)| \Big) \\ &\qquad = \sum_{n=0}^{\infty} |\zeta_{jk}; 2n\rangle \langle \zeta_{jk}; 2n|. \end{aligned} \tag{6.28}$$

A similar construction of the projection operator on the odd subspace follows from (6.12) and (6.27):

$$\begin{aligned} &\int d^2\alpha\, \rho^2 \left(I_{\frac{p}{2}+j-1}(\rho^2) + I_{\frac{p}{2}+j}(\rho^2) \right) K_{\frac{p}{2}+j}(\rho^2) \Big(|\psi_{jk}(\alpha)\rangle \langle \psi_{jk}(\alpha)| - |\psi_{jk}(\alpha)\rangle \langle \psi_{jk}(-\alpha)| \Big) \\ &\qquad = \sum_{n=0}^{\infty} |\zeta_{jk}; 2n+1\rangle \langle \zeta_{jk}; 2n+1|. \end{aligned} \tag{6.29}$$

The decomposition of unity on the subspace $\mathcal{V}_{\langle j \rangle,k}$ now emerges from simultaneous use of (6.28) and (6.29):

$$\begin{aligned} &\int d^2\alpha\, \rho^2 \left(I_{\frac{p}{2}+j-1}(\rho^2) + I_{\frac{p}{2}+j}(\rho^2) \right) \Big(\left(K_{\frac{p}{2}+j-1}(\rho^2) + K_{\frac{p}{2}+j}(\rho^2) \right) |\psi_{jk}(\alpha)\rangle \langle \psi_{jk}(\alpha)| \\ &\qquad + \left(K_{\frac{p}{2}+j-1}(\rho^2) - K_{\frac{p}{2}+j}(\rho^2) \right) |\psi_{jk}(\alpha)\rangle \langle \psi_{jk}(-\alpha)| \Big) = \sum_{n=0}^{\infty} |\zeta_{jk}; n\rangle\langle \zeta_{jk}; n| = \mathbb{I}. \end{aligned} \tag{6.30}$$

As remarked earlier, the decomposition given above includes off-diagonal terms of the density operator. The nondiagonal nature of the representation disappears if we express the density operators *via* the cat-type $|\psi_{jk}(\alpha)\rangle_{\pm}$ states introduced in (6.6):

$$\begin{aligned} &\int \frac{\rho\, d\rho\, d\theta}{\pi}\, \rho^2 \Big(I_{\frac{p}{2}+j-1}(\rho^2)\, K_{\frac{p}{2}+j-1}(\rho^2)\, |\psi_{jk}(\alpha)\rangle_{+\,+}\langle \psi_{jk}(\alpha)| \\ &\qquad + I_{\frac{p}{2}+j}(\rho^2)\, K_{\frac{p}{2}+j}(\rho^2)\, |\psi_{jk}(\alpha)\rangle_{-\,-}\langle \psi_{jk}(\alpha)| \Big) = \mathbb{I}. \end{aligned} \tag{6.31}$$

Using the integral representations of the Bessel functions

$$\begin{aligned} I_\nu(z) &= \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma\left(\nu + \frac{1}{2}\right)\, \Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} (1 - t^2)^{\nu - \frac{1}{2}}\, \cosh(zt)\, dt, \qquad \nu + \frac{1}{2} > 0, \\ K_\nu(2z) &= \int_0^\infty \exp(-z \exp t)\, \exp(-z \exp(-t))\, \cosh(\nu\, t)\, dt \end{aligned} \tag{6.32}$$

we conclude that the weight function of the measure in the decomposition of the unity given via the cat-type states in (6.31) is *positive definite* for the domain $p > 1$.

7 Conclusion

To summerize, we obtained the coherent state representations of the $\mathfrak{osp}(1|4)$ algebra by constructing the eigenstates of the paraboson operator b_1^-. In the subspace $\mathcal{V}_{\langle j \rangle,k}$ the coherent state vectors $|\psi_{jk}(\alpha)\rangle$ provide a decomposition of unity with an explicitly known weight function. When expressed via the cat-type states $|\psi_{jk}(\alpha)\rangle_{\pm}$ this measure assumes a positive definite form for the

range of the order parameter $p > 1$. In addition we have produced the bicoherent states $\Psi_{jl}(\alpha, \beta)$ which are eigenstates of the mutually commuting operators b_1^- and $(b_2^-)^2$. These states live on the subspace $\oplus_m \mathcal{V}_{\langle j\rangle,k+2m}$, and their completeness on this subspace may be investigated by using the inverse Mellin transform method followed here. We hope to discuss this result elsewhere.

We conclude the paper with certain pointers towards further developments along the present lines. It is known that q-deformed parafermions play a crucial role in understanding the noncommutative space of the fuzzy torus. Similarly a q-deformed analog of the n-mode paraboson algebra may be the underlying feature of a class of fuzzy superspaces. An extension of the coherent states presented here is likely to provide a star product structure for such noncommutative superspaces. Lastly, a coordinate representation of the trilinear commutation relation of the n-mode parabosons is likely to be of significance. In view of the close affinity of the Calogero model with the single mode paraboson, such coordinate representations are likely to enhance our understanding of the correspondingly related n-body quantum integrable Hamiltonian. The bicoherent states and the matrix elements of the b_2^- operator constructed in sections 4 and 5, respectively, should help us in the description of these Hamiltonians.

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